

# ON THE $k$ -REGULARITY OF THE $k$ -ADIC VALUATION OF LUCAS SEQUENCES

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**ABSTRACT.** For integers  $k \geq 2$  and  $n \neq 0$ , let  $\nu_k(n)$  denotes the greatest nonnegative integer  $e$  such that  $k^e$  divides  $n$ . Moreover, let  $(u_n)_{n \geq 0}$  be a nondegenerate Lucas sequence satisfying  $u_0 = 0$ ,  $u_1 = 1$ , and  $u_{n+2} = au_{n+1} + bu_n$ , for some integers  $a$  and  $b$ . Shu and Yao showed that for any prime number  $p$  the sequence  $\nu_p(u_{n+1})_{n \geq 0}$  is  $p$ -regular, while Medina and Rowland found the rank of  $\nu_p(F_{n+1})_{n \geq 0}$ , where  $F_n$  is the  $n$ -th Fibonacci number.

We prove that if  $k$  and  $b$  are relatively prime then  $\nu_k(u_{n+1})_{n \geq 0}$  is a  $k$ -regular sequence, and for  $k$  a prime number we also determine its rank. Furthermore, as an intermediate result, we give explicit formulas for  $\nu_k(u_n)$ , generalizing a previous theorem of Sanna concerning  $p$ -adic valuations of Lucas sequences.

## 1. INTRODUCTION

For integers  $k \geq 2$  and  $n \neq 0$ , let  $\nu_k(n)$  denotes the greatest nonnegative integer  $e$  such that  $k^e$  divides  $n$ . In particular, if  $k = p$  is a prime number then  $\nu_p(\cdot)$  is the usual  $p$ -adic valuation. We shall refer to  $\nu_k(\cdot)$  as the  $k$ -adic valuation, although, strictly speaking, for composite  $k$  this is not a “valuation” in the algebraic sense of the term, since it is not true that  $\nu_k(mn) = \nu_k(m) + \nu_k(n)$  for all integers  $m, n \neq 0$ .

Valuations of sequences with combinatorial meanings have been studied by several authors (see, e.g., [4, 6, 7, 8, 9, 10, 12, 14, 15, 18]). To this end, an important role is played by the family of  $k$ -regular sequences, which were first introduced and studied by Allouche and Shallit [1, 2, 3] with the aim of generalizing the concept of automatic sequences.

Given a sequence of integers  $s(n)_{n \geq 0}$ , its  $k$ -kernel is defined as the set of subsequences

$$\ker_k(s(n)_{n \geq 0}) := \{s(k^e n + i)_{n \geq 0} : e \geq 0, 0 \leq i < k^e\}.$$

Then  $s(n)_{n \geq 0}$  is said to be  $k$ -regular if the  $\mathbb{Z}$ -module  $\langle \ker_k(s(n)_{n \geq 0}) \rangle$  generated by its  $k$ -kernel is finitely generated. In such a case, the *rank* of  $s(n)_{n \geq 0}$  is the rank of this  $\mathbb{Z}$ -module.

Allouche and Shallit provided many examples of regular sequences. In particular, they showed that the sequence of  $p$ -adic valuations of factorials  $\nu_p(n!)_{n \geq 0}$  is  $p$ -regular [1, Example 9], and that the sequence of 3-adic valuations of sums of central binomial coefficients

$$\nu_3 \left( \sum_{i=0}^n \binom{2i}{i} \right)_{n \geq 0}$$

is 3-regular [1, Example 23]. Furthermore, for any polynomial  $f(x) \in \mathbb{Q}[x]$  with no roots in the natural numbers, Bell [5] proved that the sequence  $\nu_p(f(n))_{n \geq 0}$  is  $p$ -regular if and only if  $f(x)$  factors as a product of linear polynomials in  $\mathbb{Q}[x]$  times a polynomial with no root in the  $p$ -adic integers.

Fix two integers  $a$  and  $b$ , and let  $(u_n)_{n \geq 0}$  be the Lucas sequence of characteristic polynomial  $f(x) = x^2 - ax - b$ , i.e.,  $(u_n)_{n \geq 0}$  is the integral sequence satisfying  $u_0 = 0$ ,  $u_1 = 1$ , and  $u_{n+2} = au_{n+1} + bu_n$ , for each integer  $n \geq 0$ . Assume also that  $(u_n)_{n \geq 0}$  is nondegenerate, i.e.,  $b \neq 0$  and the ratio  $\alpha/\beta$  of the two roots  $\alpha, \beta \in \mathbb{C}$  of  $f(x)$  is not a root of unity.

Using  $p$ -adic analysis, Shu and Yao [16, Corollary 1] proved the following result.

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**Theorem 1.1.** *For each prime number  $p$ , the sequence  $\nu_p(u_{n+1})_{n \geq 0}$  is  $p$ -regular.*

In the special case  $a = b = 1$ , i.e., when  $(u_n)_{n \geq 0}$  is the sequence of Fibonacci numbers  $(F_n)_{n \geq 0}$ , Medina and Rowland [11] gave an algebraic proof of Theorem 1.1 and also determined the rank of  $\nu_p(F_{n+1})_{n \geq 0}$ . Their result is the following.

**Theorem 1.2.** *For each prime number  $p$  the sequence  $\nu_p(F_{n+1})_{n \geq 0}$  is  $p$ -regular. Precisely, for  $p \neq 2, 5$  the rank of  $\nu_p(F_{n+1})_{n \geq 0}$  is  $\alpha(p) + 1$ , where  $\alpha(p)$  is the least positive integer such that  $p \mid F_{\alpha(p)}$ , while for  $p = 2$  the rank is 5, and for  $p = 5$  the rank is 2.*

In this paper, we extend both Theorem 1.1 and Theorem 1.2 to  $k$ -adic valuations with  $k$  relatively prime to  $b$ . Let  $\Delta := a^2 + 4b$  be the discriminant of  $f(x)$ . Also, for each positive integer  $m$  relatively prime to  $b$  let  $\tau(m)$  denotes the *rank of apparition* of  $m$  in  $(u_n)_{n \geq 0}$ , i.e., the least positive integer  $n$  such that  $m \mid u_n$  (which is well-defined, see, e.g., [13]).

Our first two results are the following.

**Theorem 1.3.** *If  $k \geq 2$  is an integer relatively prime to  $b$ , then the sequence  $\nu_k(u_{n+1})_{n \geq 0}$  is  $k$ -regular.*

**Theorem 1.4.** *Let  $p$  be a prime number not dividing  $b$ , and let  $r$  be the rank of  $\nu_p(u_{n+1})_{n \geq 0}$ .*

- *If  $p \mid \Delta$  then:*
  - $r = 2$  if  $p \in \{2, 3\}$  and  $\nu_p(u_p) = 1$ , or if  $p \geq 5$ ;
  - $r = 3$  if  $p \in \{2, 3\}$  and  $\nu_p(u_p) \neq 1$ .
- *If  $p \nmid \Delta$  then:*
  - $r = 5$  if  $p = 2$  and  $\nu_2(u_6) \neq \nu_2(u_3) + 1$ ;
  - $r = \tau(p) + 1$  if  $p > 2$ , or if  $p = 2$  and  $\nu_2(u_6) = \nu_2(u_3) + 1$ .

Note that Theorem 1.2 follows easily from our Theorem 1.4, since in the case of Fibonacci numbers  $b = 1$ ,  $\Delta = 5$ ,  $\nu_2(F_3) = 1$ ,  $\nu_2(F_6) = 3$ , and  $\tau(p) = \alpha(p)$ .

As a preliminary step in the proof of Theorem 1.3, we obtain some formulas for the  $k$ -adic valuation  $\nu_k(u_n)$ , which generalize a previous result of the second author. Precisely, Sanna [15] proved the following formulas for the  $p$ -adic valuation of  $u_n$ .

**Theorem 1.5.** *If  $p$  is a prime number such that  $p \nmid b$ , then*

$$\nu_p(u_n) = \begin{cases} \nu_p(n) + \varrho_p(n) & \text{if } \tau(p) \mid n, \\ 0 & \text{if } \tau(p) \nmid n, \end{cases}$$

for each positive integer  $n$ , where

$$\varrho_2(n) := \begin{cases} \nu_2(u_3) & \text{if } 2 \nmid \Delta, 2 \nmid n, \\ \nu_2(u_6) - 1 & \text{if } 2 \nmid \Delta, 2 \mid n, \\ \nu_2(u_2) - 1 & \text{if } 2 \mid \Delta, \end{cases}$$

and

$$\varrho_p(n) = \varrho_p := \begin{cases} \nu_p(u_{\tau(p)}) & \text{if } p \nmid \Delta, \\ \nu_3(u_3) - 1 & \text{if } p \mid \Delta, p = 3, \\ 0 & \text{if } p \mid \Delta, p \geq 5, \end{cases}$$

for  $p \geq 3$ .

Actually, Sanna's result [15, Theorem 1.5] is slightly different but it quickly turns out to be equivalent to Theorem 1.5 using [15, Lemma 2.1(v), Lemma 3.1, and Lemma 3.2]. Furthermore, in Sanna's paper it is assumed  $\gcd(a, b) = 1$ , but the proof of [15, Theorem 1.5] works exactly in the same way also for  $\gcd(a, b) \neq 1$ .

From now on, let  $k = p_1^{a_1} \dots p_h^{a_h}$  be the prime factorization of  $k$ , where  $p_1 < \dots < p_h$  are prime numbers and  $a_1, \dots, a_h$  are positive integers.

We prove the following generalization of Theorem 1.5.

**Theorem 1.6.** *If  $k \geq 2$  is an integer relatively prime to  $b$ , then*

$$\nu_k(u_n) = \begin{cases} \nu_k(c_k(n)n) & \text{if } \tau(p_1 \cdots p_h) \mid n, \\ 0 & \text{if } \tau(p_1 \cdots p_h) \nmid n, \end{cases}$$

for any positive integer  $n$ , where

$$c_k(n) := \prod_{i=1}^h p_i^{\varrho_{p_i}(n)}.$$

Note that Theorem 1.6 is indeed a generalization of Theorem 1.5. In fact, if  $k = p$  is a prime number then obviously

$$\nu_p(c_p(n)n) = \nu_p(p^{\varrho_p(n)}n) = \nu_p(n) + \varrho_p(n),$$

for each positive integer  $n$ .

## 2. PRELIMINARIES

In this section we collect some preliminary facts needed to prove the results of this paper. We begin with some lemmas on  $k$ -regular sequences.

**Lemma 2.1.** *If  $s(n)_{n \geq 0}$  and  $t(n)_{n \geq 0}$  are two  $k$ -regular sequences, then  $(s(n) + t(n))_{n \geq 0}$  and  $s(n)t(n)_{n \geq 0}$  are  $k$ -regular too. Precisely, if  $A$  is a finite set of generators of  $\langle \ker_k(s(n)_{n \geq 0}) \rangle$  and  $B$  is a finite set of generators of  $\langle \ker_k(t(n)_{n \geq 0}) \rangle$ , then  $A \cup B$  is a set of generators of  $\langle \ker_k((s(n) + t(n))_{n \geq 0}) \rangle$ .*

*Proof.* See [1, Theorem 2.5]. □

**Lemma 2.2.** *If  $s(n)_{n \geq 0}$  is a  $k$ -regular sequence, then for any integers  $c \geq 1$  and  $d \geq 0$  the subsequence  $s(cn + d)_{n \geq 0}$  is  $k$ -regular.*

*Proof.* See [1, Theorem 2.6]. □

**Lemma 2.3.** *Any periodic sequence is  $k$ -regular.*

*Proof.* An ultimately periodic sequence is  $k$ -automatic for all  $k \geq 2$ , see [2, Theorem 5.4.2]. A  $k$ -automatic sequence is  $k$ -regular, see [1, Theorem 1.2]. □

**Lemma 2.4.** *Let  $s(n)_{n \geq 0}$  be a sequence of integers. If there exist some*

$$(1) \quad s_1 = s, s_2, \dots, s_r \in \langle \ker_k(s(n)_{n \geq 0}) \rangle$$

*such that the sequences  $s_j(kn + i)_{n \geq 0}$ , with  $0 \leq i < k$  and  $1 \leq j \leq r$ , are  $\mathbb{Z}$ -linear combinations of  $s_1, \dots, s_r$ , then  $s(n)_{n \geq 0}$  is  $k$ -regular and  $\langle \ker_k(s(n)_{n \geq 0}) \rangle$  is generated by  $s_1, \dots, s_r$ .*

*Proof.* It is sufficient to prove that  $s(k^e n + i)_{n \geq 0} \in \langle s_1, \dots, s_r \rangle$  for all integers  $e \geq 0$  and  $0 \leq i < k^e$ . In fact, this claim implies that  $\langle \ker_k(s(n)_{n \geq 0}) \rangle \subseteq \langle s_1, \dots, s_r \rangle$ , while by (1) we have  $\langle s_1, \dots, s_r \rangle \subseteq \langle \ker_k(s(n)_{n \geq 0}) \rangle$ , hence  $\langle \ker_k(s(n)_{n \geq 0}) \rangle = \langle s_1, \dots, s_r \rangle$  and so  $s(n)_{n \geq 0}$  is  $k$ -regular. We proceed by induction on  $e$ . For  $e = 0$  the claim is obvious since  $s = s_1$ . Suppose  $e \geq 1$  and that the claim holds for  $e - 1$ . We have  $i = k^{e-1}j + i'$ , for some integers  $0 \leq j < k$  and  $0 \leq i' < k^{e-1}$ . Therefore, by the induction hypothesis,

$$\begin{aligned} s(k^e n + i)_{n \geq 0} &= s(k^{e-1}(kn + j) + i')_{n \geq 0} \\ &\in \langle s_1(kn + j)_{n \geq 0}, \dots, s_r(kn + j)_{n \geq 0} \rangle \\ &\subseteq \langle s_1, \dots, s_r \rangle, \end{aligned}$$

and the claim follows. □

The next lemma is well-known, we give the proof just for completeness.

**Lemma 2.5.** *The sequence  $\nu_k(n + 1)_{n \geq 0}$  is  $k$ -regular of rank 2. Indeed,  $\langle \ker_k(\nu_k(n + 1)_{n \geq 0}) \rangle$  is generated by  $\nu_k(n + 1)_{n \geq 0}$  and the constant sequence  $(1)_{n \geq 0}$ .*

*Proof.* For all nonnegative integers  $n$  and  $i < k$  we have

$$\nu_k(kn + i + 1) = \begin{cases} 1 + \nu_k(n + 1) & \text{if } i = k - 1, \\ 0 & \text{if } i < k - 1. \end{cases}$$

Therefore, putting  $s_1 = \nu_k(n + 1)_{n \geq 0}$  and  $s_2 = (1 + \nu_k(n + 1))_{n \geq 0}$  in Lemma 2.4, we obtain that  $\langle \ker_k(\nu_k(n + 1)_{n \geq 0}) \rangle$  is generated by  $\nu_k(n + 1)_{n \geq 0}$  and  $(1 + \nu_k(n + 1))_{n \geq 0}$ , hence it is also generated by  $\nu_k(n + 1)_{n \geq 0}$  and  $(1)_{n \geq 0}$ , which are obviously linearly independent. Thus  $\nu_k(n + 1)_{n \geq 0}$  is  $k$ -regular of rank 2.  $\square$

Now we state a lemma that relates the  $k$ -adic valuation of an integer with its  $p_i$ -adic valuations. The proof is quite straightforward and we leave it to the reader.

**Lemma 2.6.** *We have*

$$\nu_k(m) = \min_{i=1, \dots, h} \left\lfloor \frac{\nu_{p_i}(m)}{a_i} \right\rfloor,$$

for any integer  $m \neq 0$ .

We conclude this section with two lemmas on the rank of apparition  $\tau(n)$ .

**Lemma 2.7.** *For each prime number  $p$  not dividing  $b$ ,*

$$\tau(p) \mid p - (-1)^{p-1} \left( \frac{\Delta}{p} \right),$$

where  $\left( \frac{\cdot}{p} \right)$  denotes the Legendre symbol. In particular, if  $p \mid \Delta$  then  $\tau(p) = p$ .

*Proof.* The case  $p = 2$  is easy. For  $p > 2$  see [17, Lemma 1].  $\square$

**Lemma 2.8.** *If  $m$  and  $n$  are two positive integers relatively prime to  $b$ , then*

$$\tau(\text{lcm}(m, n)) = \text{lcm}(\tau(m), \tau(n)).$$

*Proof.* See [13, Theorem 1(a)].  $\square$

### 3. PROOF OF THEOREM 1.6

Thanks to Lemma 2.6, we know that

$$(2) \quad \nu_k(u_n) = \min_{i=1, \dots, h} \left\lfloor \frac{\nu_{p_i}(u_n)}{a_i} \right\rfloor.$$

Moreover, from Lemma 2.8 it follows that

$$\tau(p_1 \cdots p_h) = \text{lcm}\{\tau(p_1), \dots, \tau(p_h)\}.$$

Therefore, on the one hand, if  $\tau(p_1 \cdots p_h) \nmid n$  then  $\tau(p_i) \nmid n$  for some  $i \in \{1, \dots, h\}$ , so that by Theorem 1.5 we have  $\nu_{p_i}(u_n) = 0$ , which together with (2) implies  $\nu_k(u_n) = 0$ , as claimed.

On the other hand, if  $\tau(p_1 \cdots p_h) \mid n$  then  $\tau(p_i) \mid n$  for  $i = 1, \dots, h$ . Hence, from (2), Theorem 1.5, and Lemma 2.6, we obtain

$$\nu_k(u_n) = \min_{i=1, \dots, h} \left\lfloor \frac{\nu_{p_i}(n) + \varrho_{p_i}(n)}{a_i} \right\rfloor = \min_{i=1, \dots, h} \left\lfloor \frac{\nu_{p_i}(c_k(n)n)}{a_i} \right\rfloor = \nu_k(c_k(n)n),$$

so that the proof is complete.

## 4. PROOF OF THEOREM 1.3

Clearly, if  $k$  is fixed, then  $c_k(n)$  depends only of the parity of  $n$ . Thus it follows easily from Theorem 1.6 that

$$(3) \quad \nu_k(u_{n+1}) = \nu_k(c_k(1)(n+1))s(n) + \nu_k(c_k(2)(n+1))t(n),$$

for each integer  $n \geq 0$ , where the sequences  $s(n)_{n \geq 0}$  and  $t(n)_{n \geq 0}$  are defined by

$$s(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \mid n+1, 2 \nmid n+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \mid n+1, 2 \mid n+1, \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, by Lemma 2.5 and Lemma 2.2, we know that both  $\nu_k(c_k(1)(n+1))_{n \geq 0}$  and  $\nu_k(c_k(2)(n+1))_{n \geq 0}$  are  $k$ -regular sequences. On the other hand, by Lemma 2.3, also the sequences  $s(n)_{n \geq 0}$  and  $t(n)_{n \geq 0}$  are  $k$ -regular, since obviously they are periodic.

In conclusion, thanks to (3) and Lemma 2.1, we obtain that  $\nu_k(u_{n+1})_{n \geq 0}$  is a  $k$ -regular sequence.

## 5. PROOF OF THEOREM 1.4

First, suppose that  $p \mid \Delta$ . By Lemma 2.7 we have  $\tau(p) = p$ . Moreover, it is clear that  $\varrho_p(n) = \varrho_p$  does not depend on  $n$ . As a consequence, from Theorem 1.5 it follows easily that

$$(4) \quad \nu_p(u_{n+1}) = \nu_p(n+1) + s(n),$$

for any integer  $n \geq 0$ , where the sequence  $s(n)_{n \geq 0}$  is defined by

$$s(n) := \begin{cases} \varrho_p & \text{if } n+1 \equiv 0 \pmod{p}, \\ 0 & \text{if } n+1 \not\equiv 0 \pmod{p}. \end{cases}$$

On the one hand, if  $p \in \{2, 3\}$  and  $\nu_p(u_p) = 1$ , or if  $p \geq 5$ , then  $\varrho_p = 0$ . Thus  $s(n)_{n \geq 0}$  is identically zero and it follows by (4) and Lemma 2.5 that  $r = 2$ . On the other hand, if  $p \in \{2, 3\}$  and  $\nu_p(u_p) \neq 1$ , then  $\varrho_p \neq 0$ . Moreover, for  $i = 0, \dots, p-1$  we have

$$s(pn+i) = \begin{cases} \varrho_p & \text{if } i = p-1, \\ 0 & \text{if } i \neq p-1, \end{cases}$$

hence from Lemma 2.4 it follows that  $s(n)_{n \geq 0}$  is  $p$ -regular and that  $\langle \ker_p(s(n)_{n \geq 0}) \rangle$  is generated by  $s(n)_{n \geq 0}$  and  $(\varrho_p)_{n \geq 0}$ . Therefore, by (4), Lemma 2.5, and Lemma 2.1, we obtain that  $\nu_p(u_{n+1})_{n \geq 0}$  is a  $p$ -regular sequence and that  $\langle \ker_p(\nu_p(u_{n+1})_{n \geq 0}) \rangle$  is generated by  $\nu_p(n+1)_{n \geq 0}$ ,  $s(n)_{n \geq 0}$ , and  $(1)_{n \geq 0}$ , which are clearly linearly independent, hence  $r = 3$ .

Now suppose  $p \nmid \Delta$ . By Lemma 2.7, we know that  $p \equiv \varepsilon \pmod{\tau(p)}$ , for some  $\varepsilon \in \{-1, +1\}$ . Furthermore, if  $p = 2$  then it follows easily that  $\tau(2) = 3$ . As a consequence, from Theorem 1.5 we obtain that

$$(5) \quad \nu_p(u_{n+1}) = s(n) + t(n),$$

for any integer  $n \geq 0$ , where the sequences  $s(n)_{n \geq 0}$  and  $t(n)_{n \geq 0}$  are defined by

$$s(n) := \begin{cases} \nu_p(n+1) + v & \text{if } n+1 \equiv 0 \pmod{\tau(p)} \\ 0 & \text{if } n+1 \not\equiv 0 \pmod{\tau(p)}, \end{cases}$$

with  $v := \nu_p(u_{\tau(p)})$ , and

$$t(n) := \begin{cases} \nu_2(u_6) - \nu_2(u_3) - 1 & \text{if } p = 2, n+1 \equiv 0 \pmod{6}, \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that  $s(n)_{n \geq 0}$  is a  $p$ -regular sequence of rank  $\tau(p) + 1$ . Let us define the sequences  $s_j(n)_{n \geq 0}$ , for  $j = 0, \dots, \tau(p) - 1$ , by

$$s_j(n) := \begin{cases} 1 & \text{if } n + j + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } n + j + 1 \not\equiv 0 \pmod{\tau(p)}. \end{cases}$$

On the one hand, for  $i = 0, \dots, p - 2$  we have

$$\begin{aligned} s(pn + i) &= \begin{cases} \nu_p(pn + i + 1) + v & \text{if } pn + i + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } pn + i + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= \begin{cases} v & \text{if } \varepsilon n + i + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } \varepsilon n + i + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= \begin{cases} v & \text{if } n + (\varepsilon(i + 1) - 1) + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } n + (\varepsilon(i + 1) - 1) + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= v \cdot s_{(\varepsilon(i+1)-1) \bmod \tau(p)}(n), \end{aligned}$$

since  $p \nmid i + 1$  and consequently  $\nu_p(pn + i + 1) = 0$ .

On the other hand,

$$\begin{aligned} (6) \quad s(pn + p - 1) &= \begin{cases} \nu_p(pn + p) + v & \text{if } p(n + 1) \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } p(n + 1) \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= \begin{cases} \nu_p(n + 1) + v + 1 & \text{if } n + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } n + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= s(n) + s_0(n), \end{aligned}$$

since  $\nu_p(pn + p) = \nu_p(n + 1) + 1$  and  $\gcd(p, \tau(p)) = 1$ .

Furthermore, for  $i = 0, \dots, p - 1$  and  $j = 0, \dots, \tau(p) - 1$ ,

$$\begin{aligned} s_j(pn + i) &= \begin{cases} 1 & \text{if } pn + i + j + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } pn + i + j + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= \begin{cases} 1 & \text{if } n + (\varepsilon(i + j + 1) - 1) + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } n + (\varepsilon(i + j + 1) - 1) + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= s_{(\varepsilon(i+j+1)-1) \bmod \tau(p)}(n). \end{aligned}$$

Summarizing, the sequences  $s(pn + i)_{n \geq 0}$  and  $s_j(pn + i)_{n \geq 0}$ , for  $i = 0, \dots, p - 1$  and  $j = 0, \dots, \tau(p) - 1$ , are  $\mathbb{Z}$ -linear combinations of  $s(n)_{n \geq 0}$  and  $s_j(n)_{n \geq 0}$ .

Moreover, for  $i = 0, \dots, p^2 - 1$  we have

$$\begin{aligned} (7) \quad s_0(p^2n + i) &= \begin{cases} 1 & \text{if } p^2n + i + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } p^2n + i + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= \begin{cases} 1 & \text{if } n + i + 1 \equiv 0 \pmod{\tau(p)}, \\ 0 & \text{if } n + i + 1 \not\equiv 0 \pmod{\tau(p)}, \end{cases} \\ &= s_{i \bmod \tau(p)}(n), \end{aligned}$$

hence, by (7) and (6), it follows that

$$\begin{aligned} (8) \quad s_{i \bmod \tau(p)}(n)_{n \geq 0} &= s_0(p^2n + i)_{n \geq 0} \\ &= s(p^3n + pi + p - 1)_{n \geq 0} - s(p^2n + i)_{n \geq 0} \\ &\in \langle \ker_p(s(n)_{n \geq 0}) \rangle. \end{aligned}$$

Since  $\tau(p) \mid p - \varepsilon$ , we have

$$\tau(p) \leq p - \varepsilon \leq p + 1 < p^2,$$

hence by (8) we get that  $s_j(n)_{n \geq 0} \in \langle \ker_p(s(n)_{n \geq 0}) \rangle$ , for each  $j = 0, \dots, \tau(p) - 1$ .

Therefore, in light of Lemma 2.4, we obtain that  $s(n)_{n \geq 0}$  is a  $p$ -regular sequence and that  $\langle \ker_p(s(n)_{n \geq 0}) \rangle$  is generated by  $s(n)_{n \geq 0}$  and  $s_j(n)_{n \geq 0}$ , with  $j = 0, \dots, \tau(p) - 1$ . It is straightforward to see that these last sequences are linearly independent, hence  $s(n)_{n \geq 0}$  has rank  $\tau(p) + 1$ .

If  $p > 2$ , or if  $p = 2$  and  $\nu_2(u_6) = \nu_2(u_3) + 1$ , then  $t(n)_{n \geq 0}$  is identically zero, thus from (5) and the previous result on  $s(n)$  we find that  $r = \tau(p) + 1$ .

So it remains only to consider the case  $p = 2$  and  $\nu_2(u_6) \neq \nu_2(u_3) + 1$ . Recall that in such a case  $\tau(2) = 3$ , and put  $d := \nu_2(u_6) - \nu_2(u_3) - 1$ . Obviously, the sequence  $t(2n)_{n \geq 0}$  is identically zero, while

$$\begin{aligned} t(2n+1) &= \begin{cases} d & \text{if } 2n+2 \equiv 0 \pmod{6}, \\ 0 & \text{if } 2n+2 \not\equiv 0 \pmod{6}, \end{cases} \\ &= \begin{cases} d & \text{if } n+1 \equiv 0 \pmod{3}, \\ 0 & \text{if } n+1 \not\equiv 0 \pmod{3}, \end{cases} \\ &= d \cdot s_0(n). \end{aligned}$$

Thus, again from Lemma 2.4, we have that  $t(n)$  is a 2-regular sequence and that  $\langle \ker_p(t(n)_{n \geq 0}) \rangle$  is generated by  $t(n)_{n \geq 0}$  and  $d \cdot s_j(n)_{n \geq 0}$ , for  $j = 0, 1, 2$ .

In conclusion, by (5) and Lemma 2.1, we obtain that  $\nu_p(u_{n+1})_{n \geq 0}$  is a 2-regular sequence and that  $\langle \ker_p(\nu_p(u_{n+1})_{n \geq 0}) \rangle$  is generated by  $s(n)$ ,  $t(n)$ , and  $s_j(n)$ , for  $j = 0, 1, 2$ , which are linearly independent, hence  $r = 5$ . The proof is complete.

## 6. CONCLUDING REMARKS

It might be interesting to understand if, actually,  $\nu_k(u_{n+1})_{n \geq 0}$  is  $k$ -regular for every integer  $k \geq 2$ , so that Theorem 1.3 holds even by dropping the assumption that  $k$  and  $b$  are relatively prime. A trivial observation is that if  $k$  and  $b$  have a common prime factor  $p$  such that  $p \nmid a$ , then  $p \nmid u_n$  for all integers  $n \geq 1$ , and consequently  $\nu_k(u_{n+1})_{n \geq 0}$  is  $k$ -regular simple because it is identically zero. Thus the nontrivial case occurs when each of the prime factors of  $\gcd(b, k)$  divides  $a$ .

Another natural question is if it is possible to generalize Theorem 1.4 in order to say something about the rank of  $\nu_k(u_{n+1})_{n \geq 0}$  when  $k$  is composite. Probably, the easier cases are those when  $k$  is squarefree, or when  $k$  is a power of a prime number.

We leave these as open questions to the reader.

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